Two-dimensional small-world networks: Navigation with local information

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A navigation process is studied on a variant of the Watts-Strogatz small-world network model embedded on a square lattice. With probability p, each vertex sends out a long-range link, and the probability of the other end of this link falling on a vertex at lattice distance r away decays as $r^{-\alpha}$. Vertices on the network have knowledge of only their nearest neighbors. In a navigation process, messages are forwarded to a designated target. For $\alpha<3$ and $\alpha\neq 2$, a scaling relation is found between the average actual path length and pL, where L is the average length of the additional long range links. Given $pL>1$, a dynamic small world effect is observed, and the behavior of the scaling function at large enough pL is obtained. At $\alpha=2$ and 3, this kind of scaling breaks down, and different functions of the average actual path length are obtained. For $\alpha>3$, the average actual path length is nearly linear with network size.

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I. INTRODUCTION

The famous experiment of messages being forwarded to a target among a group of people, carried out by Milgram [1] in the 1960s, and also by Dodds *et al.* in 2003 [2] on a larger scale, reveals the existence of short paths between pairs of distant vertices in networks that appear to be regular (i.e., the small-world effect [1,3–6]). One of the important quantities that characterize this small-world effect is the average shortest path length $\overline{\langle d\rangle}$ between two vertices. On small-world networks, this value grows very slowly (relative to the case of a fully regular network) with the network size N. Recent empirical research has shown that a great variety of natural and artificial networks with their structure dominated by regularity are actually small worlds, and their average path lengths grow as $\ln N$ [7], or more slowly. (See Ref. [8] for more reviews.)

An alternative issue revealed by experiments, but less obvious, is about the realistic process of passing information on small-world networks. This kind of information navigation has been studied by Kleinberg [9]. This process goes dynamically: When a message is to be sent to a designated target, each individual forwards the message to one of its nearest neighbors (connected either by a regular link or a shortcut) based on its limited information. Without information of the whole network structure, this actual path is usually longer than the shortest one given by the topological structure. While $\overline{\langle d\rangle}$ is the average shortest path length, the average actual path length $\overline{\langle l\rangle}$ is the average number of steps required to pass messages between randomly chosen vertex pairs. As $\overline{\langle d\rangle}$ is usually referred to as the diameter of the system, in the rest of this paper $\overline{\langle l\rangle}$ shall be taken as the effective diameter, and $\overline{\langle d\rangle}\leqslant\overline{\langle l\rangle}$. It has been noted that the topology of the network may significantly affect the behavior of $\overline{\langle l\rangle}$ [4,9–12]. In other words, it may determine the efficiency of passing information.

Based on Milgram's experiment, Kleinberg studied the navigation process on a variant of the Watts-Strogatz (W-S) small-world model [3] on an $N\times N$ open regular square lattice. Each vertex sends out a long range link with probability p, and the probability of the other end falling on a vertex at Euclidean distance r_e away decays as $r_e^{-\alpha}$. Kleinberg studied $\overline{\langle l\rangle}$ when each vertex sends out one long-range link and proved a lower bound of $\overline{\langle l_{p=1}\rangle}=cN^{\beta(\alpha)}$. When $\alpha=0$, the long-range links are uniform, and $\overline{\langle l_{\alpha=0,p=1}\rangle}\propto N^{2/3}$ was obtained. de Moura *et al.* [11] studied $\overline{\langle l\rangle}$ on the D-dimensional W-S model, with $\alpha=0$ and varying p, and obtained $\overline{\langle l_{\alpha=0,p=1}\rangle}\propto N^{1/(D+1)}$, and thus $\overline{\langle l_{\alpha=0,p=1}\rangle}\propto N^{1/3}$ in the two-dimensional case. In the more recent work of Zhu *et al.* [10] on the one-dimensional case, the variance of $\overline{\langle l_{\alpha,p}\rangle}$ with both α and p was studied, and scaling relations were shown to exist. For the studies of the searching processes on other different networks, see Refs. [4,13].

In this paper, we systematically investigate the navigation process on a two-dimensional variant of the W-S network model [3,14]. We study the behavior of $\overline{\langle l\rangle}$ by first working out the scaling relations in the two-dimensional case. Our result also provides new understanding of the scaling analy-

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sis in Ref. [10]. In Sec. II, the model used here is constructed and the navigation process is described, and then the average actual path length $\overline{\langle l\rangle}$ is obtained with some approximation based on a rigorous treatment. Following that, in Sec. III, the dependence of $\overline{\langle l\rangle}$ on N, α, and p are presented based on scaling relations. Special attention is paid to the cases studied in the works of Kleinberg [9] and de Moura [11]. Our summary and discussions can be found in Sec. IV.

II. THE CONSTRUCTION OF THE NAVIGATION MODEL

Our model starts from a $N\times N$ two-dimensional square lattice. With periodic boundary condition, the lattice distance between two vertices (x,y) and (x',y') can be written in a two-dimensional fashion as

$$r_{(x,y),(x',y')} = \Delta x + \Delta y, \quad \Delta x = N/2 - \big||x-x'| - N/2\big|,$$
$$\Delta y = N/2 - \big||y-y'| - N/2\big|. \tag{1}$$

This value is actually the length of the shortest path connecting these two vertices through only regular links. To generate a small world, with probability p $(0\leqslant p\leqslant 1)$ each vertex sends out an additional link to another vertex (excluding its original nearest neighbors) [18]. If this other vertex is selected at random, then we are creating a small-world network with random shortcuts. Based on realistic considerations (for example, people tend to be brought together by similar interest, occupation, etc.), we shall add the shortcuts in a biased manner [9,14]: If the shortcut starts from vertex $i(x_i,y_i)$, the probability that vertex $j(x_j,y_j)$ is selected as the end depends on the lattice distance between them in the following way,

$$P(r_{(x,y),(x',y')}) = \frac{1}{A} r^{-\alpha}_{(x,y),(x',y')},$$

where α is a positive exponent and

$$A = \sum_{(x'',y'')\neq(x,y),(x\pm1,y),(x,y\pm1)} r^{-\alpha}_{(x,y),(x'',y'')}$$

is the normalization factor.

In the model described above, the navigation process can be simulated with the so-called "greedy" algorithm [9]: Without loss of generality, suppose the target is vertex (0,0). At each step, the current message holder, vertex $n(x_n,y_n)$, passes the message through one of its regular or long-range links. Based on its limited local information, this link is believed to bring the message the closest to the target. Based on this algorithm, the actual path length $\langle l(x_n,y_n)\rangle$ can be obtained after taking an ensemble average over all possible realizations of the network (with a set of fixed parameters, p, N, α, etc.).

In the simplest case, we suppose that each vertex has information of only the vertices that can be reached within one step, and do the calculation as the following: (1) If the current message holder is vertex (0,1), we simply have

$$\langle l(0,1)\rangle = 1. \tag{2}$$

It is the same for the other nearest neighbors of the target $(0,-1)$, (1,0), and $(-1,0)$. (2) There are eight nodes with lattice distance 2 from the target: $(0,\pm2)$, $(\pm2,0)$, $(\pm1,\pm1)$, and $(\mp1,\pm1)$. If the current message holder is, for example, (0,2), then with probability

$$W_{(0,2)\to(0,0)} = 1 - \left(1 - p\frac{2^{-\alpha}}{A}\right)^2,$$

it is directly linked to the target via one shortcut, which means the message is sent directly to the target with this probability. On the other hand, the probability that the message is forwarded along a regular bond is

$$W_{reg} = 1 - W_{(0,2)\to(0,0)}.$$

Thus

$$\langle l(0,2)\rangle = 1\times W_{(0,2)\to(0,0)} + 2\times W_{reg}. \tag{3}$$

The calculation is the same for the other seven nodes mentioned above. (3) In a general case, the message is held by vertex (x,y). $W_{(x,y)\to(x',y')}$ denotes the probability that the message is forwarded in the next step to a vertex (x',y'), which must be nearer to the target than (x,y) by at least lattice distance 2. If the message holder is not able to find a shortcut, the message will be forwarded along a regular link with probability

$$W_{reg} = 1 - \sum_{(x',y')} W_{(x,y)\to(x',y')}. \tag{4}$$

For example, if the vertex (3,2) passes the message through a regular link, it will randomly choose (2,2) or (3,1), which in the following will be denoted by (x_{reg},y_{reg}).

Now, with this set of probabilities W, we obtain

$$\langle l(x,y)\rangle = W_{(x,y)\to(0,0)} + \sum_{(x',y')} W_{(x,y)\to(x',y')}[1+\langle l(x',y')\rangle]$$
$$+ W_{reg}[1+\langle l(x_{reg},y_{reg})\rangle]. \tag{5}$$

Considering that p/A is a relatively small quantity, $W_{(x,y)\to(x',y')}$ can be expressed as [10]

$$W_{(x,y)\to(x',y')} = 2p\frac{r^{-\alpha}_{(x,y),(x',y')}}{A}, \tag{6}$$

where we have used the fact that $r_{(x,y),(x',y')}=r_{(x',y'),(x,y)}$. Then it is an easy task to obtain W_{reg} from Eq. (4).

Recall that we have the definition of the average shortest path length $\overline{\langle d\rangle}=\frac{1}{N^2(N^2-1)}\sum_{i\neq j}d_{ij}$, where d_{ij} is the length of the shortest path between vertices i and j. By contrast, the average actual path length can be defined as

$$\overline{\langle l\rangle} = \frac{1}{N^2(N^2-1)}\sum_{(x,y)\neq(0,0)}\langle l(x,y)\rangle. \tag{7}$$

Further, with vertex (0,0) being the target, we group the other nodes according to their lattice distance from the target, and in the following we shall also discuss the function $\overline{\langle l(n)\rangle}$, which for each value of n is obtained by averaging all nodes (x,y) with $r_{(x,y),(0,0)}=n$.

III. FEATURES OF THE NAVIGATION PROCESS

The average actual path length depends on multiple parameters. Here we take into consideration varying N, α, and p, but keep the range of view of each vertex limited to its nearest neighbors. Our discussion of the navigation process starts from looking for the basic scaling relations.

A scaling relation is not new in the theories of small-world effect. Actually, it plays a central role in the current theoretical framework. In 1999, Newman [15] showed that in the W-S model with uniform shortcuts the average shortest path length is a function of pN, and it sharply decreases when pN becomes larger than 1. Newman noticed that pN is simply the expected number M of long-range links. The threshold of small-world behavior is $M=pN>1$, which means the network becomes a small world when there is more than one long-range link. When the model network is generalized, this interpretation shall be generalized as well. For example, in a discussion of the scaling relations in the problem of dynamic navigation, Zhu *et al.* [10] considered inhomogeneous long-range links (the probability of linking two nodes falls when their lattice distance increases). In their study, the dynamic small world behavior is switched on when $ML'>1$, where M is the number of long-range links and L' is the average reduced link length (the average length of long range links L divided by the system size). Although they focused on different aspects (static and dynamic) of the small world effect, we can still compare these two versions of scaling relations. Because in the model studied by Newman $L'\sim 1/4$, it is consistent with the interpretation of Zhu *et al.* Actually, as we shall see below, the interpretation of Zhu *et al.* can be developed as well, when a more general model is considered.

In the Introduction we have defined $\overline{\langle l\rangle}$ as the effective diameter; in the following we will use $\overline{\langle l'\rangle}=\overline{\langle l\rangle}/N$ as the reduced effective diameter. If the network is regular, $\overline{\langle l'\rangle}$ will appear as a constant. Bearing this in mind, we first look at the results shown in Fig. 1. For each value of α (with an exception at $\alpha=2$, as will be shown below), $\overline{\langle l'\rangle}$ appears as a function of pL, where L is the average length of long-range links. Thus our study clearly supports an interpretation different from that in Ref. [10]: Instead of ML', the parameter should be pL. In the one-dimensional case, this equals ML' and is thus consistent with Ref. [10]. When $pL\ll 1$, $\overline{\langle l'\rangle}\to 0.5$ and $\overline{\langle l\rangle}\propto N$, indicating that the network is virtually regular, and when pL increases beyond 1, the system begins to show a dynamic small-world behavior.

However, we find interesting exceptions at $\alpha=2$ and $\alpha=3$. As shown in Fig. 2, at $\alpha=2$, $\exp(\overline{\langle l'\rangle}\times pL)$ is a linear function of $pL/\ln N$ for pL significantly larger than 1. Due to this extra factor of $1/\ln N$, there is no way that $\overline{\langle l'\rangle}$ can be written as a function of pL for $\alpha=2$. Because at $\alpha=2$ the system shows the shortest $\overline{\langle l\rangle}$, this is certainly a case of special interest. It becomes more curious when we notice that in one dimension there is not such an exception in the scaling analysis [10]. As shown in Fig. 3, at $\alpha=3$, $\overline{\langle l'\rangle}$ appears as a function of $pL\ln N$, and the dynamic small-world behavior is seen when this parameter exceeds 1. There is more discussion of these interesting points later.

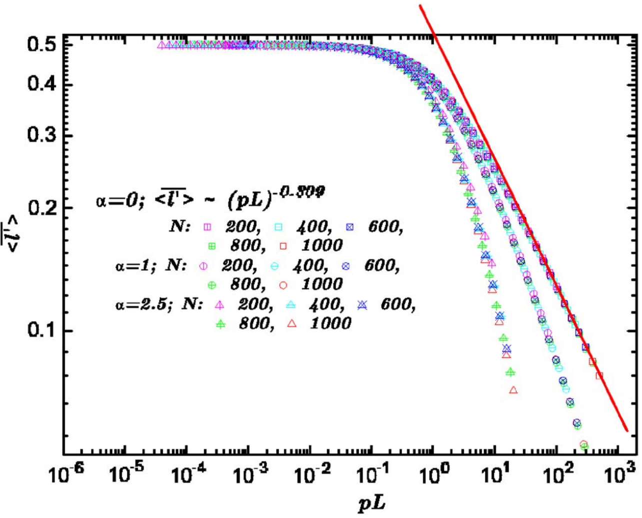

FIG. 1. (Color online) The reduced average actual path length $\overline{\langle l'\rangle}$ varies as pL for $\alpha=0,1,2.5$, where L is the average length of the additional long range links. The data collapse with each specific α contains curves with $N=200,400,600,800,1000$ respectively. On each curve with fixed N, $p=1.3^{-i}$, where $i=0,1,2,\ldots,47$ (p has the same set of values on each curve in Figs. 2 and 3). The solid line $y\sim x^{-0.309}$ is a guide to the eye.

Figures 1–3 can give us more information once we get an idea of L. For N large enough, we can use the following approximation,

$$L=\frac{\int_0^{N/2-1}dx\int_1^{N/2}(x+y)^{1-\alpha}dy}{\int_0^{N/2-1}dx\int_1^{N/2}(x+y)^{-\alpha}dy},\tag{8}$$

which gives

$$L=\frac{1-\alpha}{3-\alpha}\frac{(N-1)^{3-\alpha}-2(N/2)^{3-\alpha}+1}{(N-1)^{2-\alpha}-2(N/2)^{2-\alpha}+1}\quad(\alpha\neq 0,1,2,3),\tag{9}$$

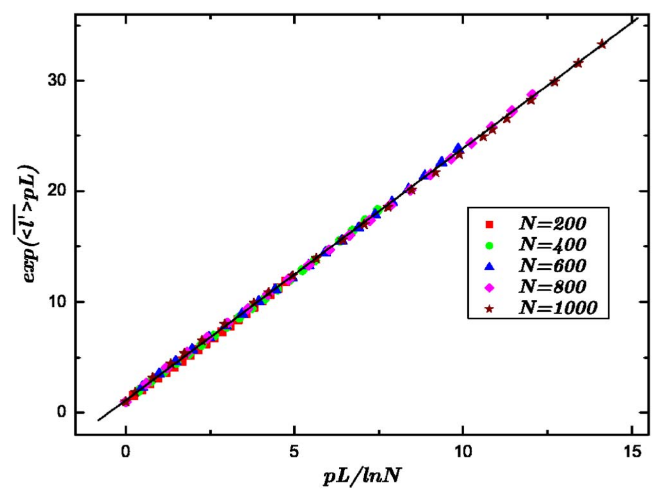

FIG. 2. (Color online) With given $\alpha=2$, plots show the relation between $\exp(\overline{\langle l'\rangle}\cdot pL)$ and pL at $N=200,400,600,800,1000$.

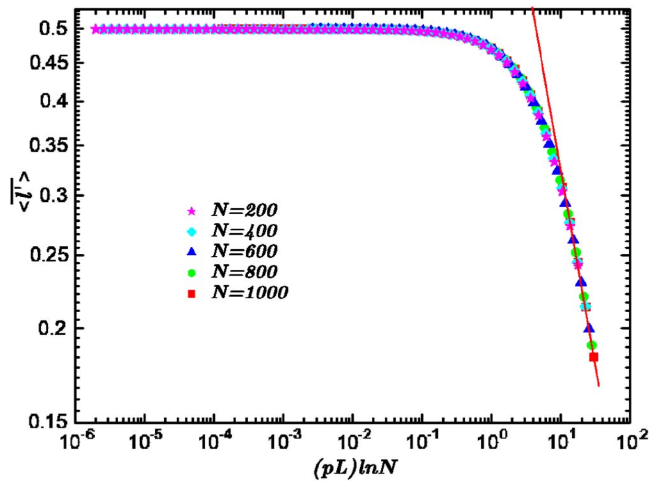

FIG. 3. (Color online) The relationship between $\overline{\langle l' \rangle}$ and $pL \ln N$ in log-log scale at $\alpha=3$ for $N=200,400,600,800,1000$.

$$L_{\alpha=0}=\frac{1}{6}\frac{N^3-2\left(\frac{N}{2}\right)^3+1}{\left(\frac{1}{2}N-1\right)^2}, \tag{10}$$

$$L_{\alpha=1}=\frac{\left(\frac{1}{2}N-1\right)^2}{(N-1)\ln(N-1)-N\ln\left(\frac{N}{2}\right)}, \tag{11}$$

$$L_{\alpha=2}=\frac{(N-1)\ln(N-1)-N\ln\left(\frac{N}{2}\right)}{2\ln\frac{N}{2}-\ln(N-1)}, \tag{12}$$

and

$$L_{\alpha=3}=\frac{2\ln\frac{N}{2}-\ln(N-1)}{\frac{1}{2(N-1)}-\frac{2}{N}+\frac{1}{2}}. \tag{13}$$

Some comments on this approximation: We have chosen the upper and lower limits for the integrals with some arbitrariness. For $\alpha \leqslant 3$ and for N very large, it gives results good enough for the later discussion. Surely this approximation fails for $\alpha > 3$, but in that region what is important is L stays finite as N goes to infinity.

From Eqs. (9)–(13), we have for large enough N

$$L_{\alpha=0}\to\frac{N}{2}, \tag{14}$$

$$L_{\alpha=1}\to\frac{N}{4(\ln 2)}, \tag{15}$$

$$L=\frac{1-\alpha}{3-\alpha}\cdot\frac{1-2^{\alpha-2}}{1-2^{\alpha-1}}N \quad (0<\alpha<1,\ 1<\alpha<2), \tag{16}$$

$$L_{\alpha=2}\to\ln 2\frac{N}{\ln N}, \tag{17}$$

$$L\to N^{3-\alpha}\frac{1-\alpha}{3-\alpha}\cdot(1-2^{\alpha-2})\to 0 \quad (2<\alpha<3), \tag{18}$$

$$L_{\alpha=3}\to 2\ln N, \tag{19}$$

$$L\to \text{finite} \quad (\alpha>3). \tag{20}$$

In the above equations, $\alpha=0,2,3$ come out as special points. $\alpha=0$ corresponds to the totally random network. Below $\alpha_c^1=D=2$, $L_{0\leqslant\alpha<2}$ is always proportional to N. In Ref. [16], the authors proposed that for $0\leqslant\alpha<\alpha_c^1$ the system is in the random network phase, and α_c^1 is the continuous phase transition point from the random network phase to the small-world phase. As α increases above $\alpha_c^2=D+1=3$, L is finite and independent of N in the limit of $N\to\infty$. As a result, the system is virtually a regular network for $\alpha>\alpha_c^2$.

Now we have the scaling relations shown in Figs. 1–3 and L as a function of N given by Eqs. (14)–(20). In the following we shall discuss the system behavior with α starting from zero.

(1) $0\leqslant\alpha<\alpha_c^1$: As shown in Fig. 1, when $pL\ll 1$, $\overline{\langle l'\rangle}\to 0.5$, indicating that the network is virtually regular. When pL increases beyond 1, the system begins to show a dynamic small-world behavior, in the sense that

$$\overline{\langle l'\rangle}\sim(pL)^{-\gamma}\sim p^{-\gamma}N^{-\gamma} \tag{21}$$

or

$$\overline{\langle l\rangle}\sim p^{-\gamma}N^{1-\gamma}, \tag{22}$$

where γ depends only on α. From the linear fit, at $\alpha=0$ we have obtained that $\gamma\approx 0.309$ and $\overline{\langle l\rangle}\sim p^{-0.309}N^{0.691}$. Note that γ is obtained from the linear fit of limited data and cannot be exact, but our result, $\beta=1-\gamma\approx 0.691$, is close to Kleinberg's result, $\beta=2/3$. (We observe that in Ref. [11], $\beta=1/3$.) As α increases, γ increases, but β remains positive as $\alpha\to\alpha_c^1$. With our present results, we are not able to give the full function of $\gamma(\alpha)$, because near $\alpha=2$, $\overline{\langle l\rangle}$ as a function of N severely deviates from a power law for relatively small N.

(2) $\alpha=\alpha_c^1=2$: At this point, when $pL/\ln N\ll 1$, $\overline{\langle l'\rangle}\to 0.5$, showing that it is a regular network. When $pL/\ln N$ increases above 1, $\exp(\overline{\langle l'\rangle}\times pL)$ turns into a linear function of $pL/\ln N$, and it means

$$\overline{\langle l'\rangle}\sim\ln(pL/\ln N)/pL. \tag{23}$$

With $L\sim\ln 2\frac{N}{\ln N}$ for $N\to\infty$, this gives

$$\overline{\langle l\rangle}\sim\frac{\ln N[\ln p+\ln(\ln 2)+\ln N-2\ln(\ln N)]}{p\ln 2}, \tag{24}$$

or, in the limit of $N\to\infty$,

$$\overline{\langle l \rangle} \sim \frac{\ln N(\ln p + \ln N)}{p}. \tag{25}$$

We have studied the case of $0 \leqslant \alpha < \alpha_c^1$, and the case of $\alpha > \alpha_c^1$ will be studied below. We shall see that at $\alpha = \alpha_c^1$ the smallest $\overline{\langle l \rangle}$ is achieved.

(3) $\alpha_c^1 < \alpha < \alpha_c^2$: Regularity dominates for $pL \ll 1$. For $pL > 1$, a dynamic small-world effect arises. Similar to the case of $0 \leqslant \alpha < \alpha_c^1$, we have once again obtained

$$\overline{\langle l \rangle} \sim p^{-\gamma} N^{1-\gamma}, \tag{26}$$

and γ tends to zero as α approaches α_c^2.

(4) $\alpha = \alpha_c^2 = 3$: When $pL \ln N \ll 1$, $\overline{\langle l' \rangle} \to 0.5$, the network also shows dominating regularity. When $pL \ln N$ greatly exceeds 1,

$$\overline{\langle l' \rangle} \sim (pL \ln N)^{-\gamma}. \tag{27}$$

Substitute $L \sim 2 \ln N$ into the above equation. Then, for large enough N, we have

$$\overline{\langle l \rangle} \sim p^{-\gamma} N (\ln N)^{-2\gamma}, \tag{28}$$

but the size of the networks used in the present study prevented us from getting an accurate estimate of γ.

(5) $\alpha > \alpha_c^2$: Since in this case L stays finite when $N \to \infty$, the system is believed to behave like a regular network. It is confirmed by the numerical calculation, which gives $\overline{\langle l \rangle}$ nearly linear as N.

IV. SUMMARY AND DISCUSSION

In this work, we investigate the navigation process on a variant of the Watts-Strogatz (W-S) model embedded in a two-dimensional square lattice with periodic boundary condition. With probability p, each vertex sends out a long-range link, and the probability that the other end of this link falls on a vertex at lattice distance r away decays as $r^{-\alpha}$. Vertices on the network have knowledge of only their nearest neighbors. In a navigation process, messages are forwarded to a designated target, and the average actual path length $\overline{\langle l \rangle}$ is obtained with varying α, p, and N.

Our result is consistent with the existence of two phase transitions at $\alpha_c^1 = D = 2$ (random network to small-world network) and $\alpha_c^2 = D + 1 = 3$ (small-world network to regular network). For $\alpha < \alpha_c^2$, and $\alpha \neq \alpha_c^1$, it is found that $\overline{\langle l' \rangle} \equiv \overline{\langle l \rangle}/N \sim f_\alpha(pL)$, where L is the average length of the additional long-range links. This develops the scaling analysis in the works of Newman [15] and Zhu *et al.* [10]. Given $pL > 1$, a dynamic small-world effect is observed, and the behavior of f_α at large enough pL gives $\overline{\langle l \rangle} \sim p^{-\gamma(\alpha)} N^{1-\gamma(\alpha)}$. When $\alpha = 0$, γ is close to $1/3$, so $\beta = 1 - \gamma \sim 2/3$, in agreement with Kleinberg's result of $\overline{\langle l \rangle} \propto N^{2/3}$ for $p = 1$. As α increases, γ increases (but stays below 1), and once α exceeds α_c^1, γ begins to decrease, and γ approaches zero as $\alpha \to \alpha_c^2$. At $\alpha = \alpha_c^1$, this kind of scaling breaks down, and $\overline{\langle l' \rangle}$ can no longer be written as a function of pL. In this case we can still get $\overline{\langle l \rangle} \sim \ln N(\ln p + \ln N)/p$ for N large enough. Note that only at $\alpha = \alpha_c^1$, $\overline{\langle l \rangle}$ grows as a polynomial of $\ln N$, and it is the closest point to the static small world effect [10]. At $\alpha = \alpha_c^2$, the scaling is $\overline{\langle l' \rangle} \sim f(pL \ln N)$, and accordingly $\overline{\langle l \rangle} \sim p^{-\gamma} N (\ln N)^{-2\gamma}$ with large enough $pL \ln N$. For $\alpha > \alpha_c^2$, $\overline{\langle l \rangle}$ is nearly linear with N.

It is reasonable that in social networks (like various other networks) the probability of connection falls as distance (in various senses, e.g., occupation) increases, and the apparently very small value of $\overline{\langle l \rangle}$ in human society [1,2] suggests human society might have its exponent α being close to α_c^1.

The great success of the idea of small world has since motivated much effort in studying various dynamic processes based on the small-world network model. The limited knowledge of the nodes of a network is an important limitation that has to be considered when studying navigation processes. Another interesting and important limitation is due to the fact that the links in a network are usually associated with "weights," as systematically studied in Ref. [17]. Further studies on link-weighted small-world models should help us gain insight in the navigation and other relevant phenomena in various artificial and natural networks, and help us design networks with higher efficiency.

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